AIAA 81-4041

Tapered Finite Elements in the Optimality Criterion Approach

G. Venkateswara Rao*
Vikram Sarabhai Space Center, Trivandrum, India

Introduction

THE optimality criterion approach for obtaining minimum mass (volume if density is constant) design of a structure subjected to a variety of constraints is well established.[1-5] In all earlier studies element detail in the form of area or thickness (constant for an element) is used as a design variable and hence the optimum structure obtained is a stepped one with discontinuities at the element boundaries.

In this Note an alternate method, in which the nodal detail is in the form of nodal area or thickness, is used as a design variable. As a consequence of this the finite elements used to model the structure will have a tapered configuration. This will enable continuity of structure at the element boundaries.

This method is applied to the problem of optimization of a cooling fin with a temperature constraint. This problem has already been solved by a continuum approach[6] and the optimality criterion approach using constant area elements.[4] The solution obtained with nodal areas as design variables, when compared to the exact solution,[6] is more accurate when compared to that obtained with element areas as design variables.[4]

Formulation

Let the cooling fin be divided into M number of linearly tapered finite elements of equal length. Then the volume of the bar V is given by

$$V=\sum_{i=1}^{n}\frac{A_i l}{p} \tag{1}$$

where n is the number of nodal points, l is the length of the element, and A_i, the design variable, is the area of the fin at node i. The value of p is 2 for $i=1$ or n and 1 for $i=2$ to $(n-1)$.

The constraint equation is

$$T_m - T_c = 0 \tag{2}$$

or

$$\{T\}^T\{Q\} - T_c = 0 \tag{3}$$

where T_m is the temperature at the node m at which the value is constrained to be T_c, $\{T\}$ is the temperature vector of the fin obtained after performing thermal analysis, and $\{Q\}$ is a vector of zeros except at the constraint degree-of-freedom where the value is unity.

Following the procedure explained in Ref. 4, the optimality criterion for the system can be obtained as

$$e_i p/l = 1 \tag{4}$$

where

$$e_i = \lambda\{T\}^T\frac{\partial[D]}{\partial A_i}\{s\} \tag{5}$$

λ is the Lagrangian multiplier, $[D]$ is the sum of the assembled conduction and convection matrices, and $\{s\}$ is the temperature vector obtained with $\{Q\}$ as the thermal load vector.

Recurrence relations for design variables A_i and λ are obtained as[4]

$$A_{j+1_i} = \sqrt{(e_i p/l)}\,A_{j_i} \tag{6}$$

and

$$\lambda_{j+1} = \sqrt{T_m/T_c}\;\lambda_j \tag{7}$$

where j is the iteration number.

It is to be noted here that $[D]$ is of the form

$$[D] = [C] + [H] \tag{8}$$

where $[C]$ is the assembled conduction matrix dependent on A_i and $[H]$ is the assembled convection matrix independent of A_i. Therefore,

$$\frac{\partial[D]}{\partial A_i} = \frac{\partial[C]}{\partial A_i} \tag{9}$$

Also, for a linearly tapered kth finite element, the element matrix $[C]_k$ can be written as,

$$[C]_k = A_k[C_1] + A_{k+1}[C_2] \tag{10}$$

and from Eq. (10), the derivative $\partial[C]/\partial A_i$ can be obtained to be used in Eq. (6).

Numerical Results

The method proposed in this note is applied to the problem of obtaining the minimum mass (volume if density is constant) design of a cooling fin (Fig. 1) subjected to a temperature constraint.

Heat is generated in the fin at a constant rate of q per unit length with convective heat loss over both the top and bottom surfaces. The end $\bar{x}=0(\bar{x}=x/L$, where x is the axial coordinate and L is the length of the fin) is maintained at a temperature of $\bar{T}=0(\bar{T}=T/T_o$, where T_o is the reference temperature) and the insulated end is constrained to a value of $\bar{T}_c=0.3$.

One-dimensional linearly tapered finite elements with cubic temperature distribution over the element are used to model the cooling fin. The elements have two nodes with $\bar{T}$ and $d\bar{T}/d\bar{x}$ as nodal degrees-of-freedom. In the present analysis the fin is idealized with five linearly tapered elements.

To initiate the iterative process an initial design of $\bar{A}_i=1$, $i=1$ to n (a fin of constant area), and $\lambda = 1$ is used. The

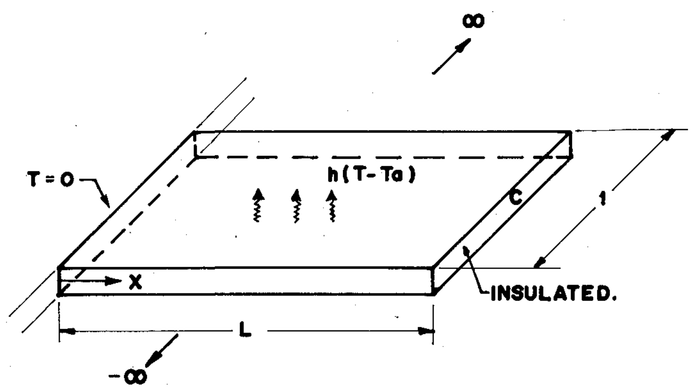

Fig. 1 Fin geometry and boundary conditions.

Received Feb. 20, 1980.

*Engineer, Structural Engineering Division.

Table 1 Optimum area ($\bar{A}$) and temperature ($\bar{T}$) distribution along length of the cooling fin – five-element solution ($\bar{T}_c = 0.3$, $N_G = 1.0$, $B_i = 0.0$)

	$\bar{A}$			$\bar{T}$		
Node	Present method	Exact solution[6]	Ref. 4	Present method	Exact solution[6]	Ref. 4
1[a]	2.2263[c]	2.2222	1.9464[d]	0.0	0.0	0.0
2	1.9915	1.9876	1.7846	0.0853	0.0853	0.0930
3	1.7281	1.7213	1.5938	0.1604	0.1650	0.1715
4	1.4092	1.4054	1.3377	0.2239	0.2241	0.2343
5	1.0271	0.9938	0.9213	0.2727	0.2732	0.2790
6[b]	0.3055	0.0		0.3000	0.3000	0.3000
$\bar{V}_{opt}$	1.4845	1.4815	1.5168			

[a]Node at $\bar{T} = 0$. [b]Node at insulated end. [c]Nodal areas of linearly tapered elements (solution at 25 interations). [d]Constant area elements (solution at 20 iterations).

Table 2 Effect of convection on minimum volume for cooling fin – five-element solution ($\bar{T}_c = 0.3$, $\bar{T}_a = 0.05$, $N_G = 1.0$)

	$\bar{V}$	
B_i	Present method	Constant area elements (Ref. 4)[c]
0.0[a]	1.4845 (25)[b]	1.5168 (1.5023)[d]
0.05	1.4564 (25)	1.4878 (1.4736)
0.2	1.3712 (30)	1.4005 (1.3870)
0.5	1.1985 (34)	1.2232 (1.2115)

[a]For $B_i = 0.0$, $\bar{V}_{exact} = 1.4815$ (Ref. 6). [b]Number of iterations to obtain convergence. [c]Results at 20 iterations. [d]Numbers in the brackets represent 20-element solution.

iterative process is terminated when the constraint temperature converges up to four significant figures.

Table 1 gives the nondimensional area distribution $\bar{A}$ ($\bar{A} = A/A_o$, where A_o is the reference area) and temperature distribution $\bar{T}$ at the six nodes of the fin for the heat generation number, $N_G = 1.0$($N_G = qL^2/KA_oT_o$, where K is the thermal conductivity of the material), without convective heat loss, i.e., the Biot number, $B_i = 0$ ($B_i = hL^2/KA_o$, where h is the convective heat-transfer coefficient). The exact solution for this case[6] and the solution obtained with constant area elements with element areas as design variables[4] are also presented in this table. Nondimensional optimum fin volume $\bar{V}_{opt}$ ($\bar{V}_{opt} = V/A_oL$, where V is the volume of the fin) is given in this table. The present solution converged in 25 iterations and the solution of Ref. 4 is given at 20 interations. It can be seen from Table 1 that the results obtained with nodal areas as design variables agree well with the exact solution, whereas those obtained with element areas as design variables are inferior and give higher fin volume compared to the present solution.

In Table 2, the effect of convective heat loss on the minimum mass design is shown. In this table, the nondimensional volume $\bar{V}$ is presented for various values of B_i (the ambient temprature $\bar{T}_a$ is taken as 0.05). $\bar{V}$ obtained from the present five-element solution is less compared to the 20-element solution of Ref. 4. Also it can be noted here that as B_i increases, the number of iterations required for convergence increases. This is because the convection matrix is independent of design variables. From this table it can be concluded that the effect of convective heat loss is to decrease the optimum volume.

Conclusion

The concept of nodal variable as a design variable in the optimality criterion approach for obtaining minimum mass design of structures is proposed. The effectiveness of this proposal is shown through an example of obtaining the minimum mass (volume if density is constant) design of a cooling fin subjected to a temperature constraint.

References

[1]Venkayya, V. B., Khot, N. S., and Berke, L., "Application of Optimality Criteria Approaches to Automated Design of Large Practical Structures," AGARD-CP-123, 1973, pp. 3-1 – 3-19.

[2]Berke, L. and Khot, N. S., "Use of Optimality Criteria Methods for Large Scale Systems," AGARD-LS-70, 1974, pp. 1-1 – 1-29.

[3]Khot, N. S., Venkayya, V. B., and Berke, L., "Optimum Design of Composite Structures with Stress and Displacement Constraints," *AIAA Journal*, Vol. 14, 1976, pp. 131-132.

[4]Venkateswara Rao, G., Shore, C. P., and Narayanaswami, R., "An Optimality Criterion for Sizing Members of Heated Structures with Temperature Constraints," NASA TN D-8525, 1977.

[5]Venkateswara Rao, G. and Narayanaswami, R., "Optimization of a Conducting Cooling Fin with a Heat Sink Using Optimality Criterion," *International Journal of Solids and Structures*, Vol. 14, 1978, pp. 787-793.

[6]Ahmadi, G. and Razani, A., "Some Optimization Problems Related to Cooling Fins," *International Journal of Heat and Mass Transfer*, Vol. 16, 1973, pp. 2369-2375.

AIAA 81-4042

Amplification of Finite-Amplitude Waves in a Radiating Gas

R. S. Singh* and V. D. Sharma†
Institute of Technology, Banaras Hindu University, Varanasi, India

Introduction

USING the theory of singular surfaces, a number of problems relating to wave propagation in diverse branches of continuum mechanics have been studied previously.[1-7] The present paper uses the singular surface theory of Thomas[6] to study the propagation of arbitrarily shaped finite-amplitude waves in a radiating gas near the optically thin limit. The gas is assumed to be perfect, optically gray, and in thermodynamic equilibrium. It is found that a

Received Feb. 15, 1980; revision received June 16, 1980.

*Lecturer, Dept. of Mechanical Engineering.
†Lecturer, Applied Mathematics Section.